

Dirac quantization of a nonminimal gauged O(3) sigma model

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The (2+1) dimensional gauged O(3) nonlinear sigma model with Chern-Simons term is canonically quantized. Furthermore, we study a nonminimal coupling in this model implemented by means of a Pauli-type term. It is shown that the set of constraints of the model is modified by the introduction of the Pauli coupling. Moreover, we found that the quantum commutator relations in the nonminimal case is independent of the Chern-Simons coefficient, in contrast to the minimal one.

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I. INTRODUCTION

The importance of O(3) nonlinear sigma model lies in theoretical and phenomenological basis. It is a fact that this theory describe classical (anti) ferromagnetic spin systems at their critical points in Euclidean space, while in the Minkowski one it delineates the long wavelength limit of quantum antiferromagnets. The model exhibits solitons, Hopf instantons and novel spin and statistics in 2+1 space-time dimensions with inclusion of the Chern-Simons term. The gauging of a nonlinear sigma model, given rise to a coupling between the scalar fields (coordinates of the target space) and the gauge fields has attracted interest from different areas. In particular soliton solutions of the gauged O(3) Chern-Simons model may be relevant in planar condensed matter systems^{1,2,3}. Recently gauged gravitating nonlinear sigma model was considered in order to obtain self-dual cosmic string solutions⁴.

The canonical quantization of nonlinear gauged sigma models has been studied by some authors in the context of the CP^1 sigma model. Indeed this question was considered in connection with fractional spin by Panigrahi and collaborators^{5,6} and later by Han⁷. On the other hand, the canonical quantization of the O(3) nonlinear sigma model with the Hopf term was treated by Bowick *et al.*⁸ in order to compute the angular momentum and establish the relation of a fractional spin and the coefficient of the Hopf term.

In this work we investigate the soliton sector of the O(3) nonlinear sigma model coupled to an Abelian gauge field through a nonminimal term and in the presence of the Chern-Simons term⁹, using the Dirac formalism for constrained systems¹⁰. The nonminimal term could be interpreted as a generalization of the Pauli coupling, *i. e.*, an anomalous magnetic moment. It is a specific feature of (2+1) dimensions that the Pauli coupling exists not only for spinning particles, but for scalars ones too^{11,12,13,14}.

II. THE MINIMAL GAUGED O(3) NONLINEAR SIGMA MODEL

Let us begin by considering the **minimal** gauged O(3) nonlinear sigma model in a covariant gauge-fixed form. Models with an explicit gauge dependence may be suitable; for instance α in the Nakanishi-Lautrup formalism:

$$L = \frac{1}{2} D_\mu \phi \cdot D^\mu \phi + \frac{\kappa}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - A_\mu \partial^\mu b + \frac{\alpha}{2} b^2, \quad (1)$$

where

$$D_\mu \phi = \partial_\mu \phi + e A_\mu \mathbf{n} \times \phi, \quad (2)$$

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is the minimal gauge covariant derivative and the second term is the Abelian Chern-Simons term. The addition of this term to the $O(3)$ sigma model makes the vector field propagate even at the classical level. The multiplier field b is introduced in order to implement the Lorentz covariant gauge-fixed condition. The canonical momenta conjugate to the fields $A_0, A_k, b, \phi_1, \phi_2, \phi_3$ respectively, are given by $(i, j, k = 1, 2)$

$$\pi_0 = 0, \quad (3)$$

$$\pi^k = \frac{\kappa}{2} \epsilon^{ki} A_i, \quad (4)$$

$$\pi_b = -A_0, \quad (5)$$

$$\Pi_1 = \dot{\phi}_1 - eA_0\phi_2, \quad (6)$$

$$\Pi_2 = \dot{\phi}_2 + eA_0\phi_1, \quad (7)$$

$$\Pi_3 = \dot{\phi}_3. \quad (8)$$

Following the Dirac procedure for constrained systems, non-derivative canonically conjugate momenta are classified as primary constraints. Therefore, our primary constraints set is

$$V_1 = \pi_0 \approx 0, \quad (9)$$

$$V_2 = \phi \cdot \phi - 1 \approx 0 \quad (10)$$

$$V_3 = \pi_1 + \frac{\kappa}{2} A_2 \approx 0, \quad (11)$$

$$V_4 = \pi_2 - \frac{\kappa}{2} A_1 \approx 0, \quad (12)$$

$$V_5 = \pi_b + A_0 \approx 0. \quad (13)$$

So these constraints leads us to the canonical Hamiltonian

$$H_c = \pi_0 \dot{A}_0 + \pi^k \dot{A}_k + \pi_b \dot{b} + \Pi_j \dot{\phi}_j - L, \quad (14)$$

which can be put in the form

$$\begin{aligned} H_C = & -\frac{1}{2} D^j \phi \cdot D_j \phi - \frac{\mathbf{\Pi} \cdot \mathbf{\Pi}}{2} + eA_0 (\phi_2 \Pi_1 - \phi_1 \Pi_2) \\ & - \kappa \epsilon^{ij} A_0 \partial_i A_j + A_j \partial^j b - \frac{\alpha}{2} b^2. \end{aligned} \quad (15)$$

Preservation of the primary constraints (9), (10), (11), (12) and (13), yield the consistent condition

$$\dot{V}_j \approx 0 \quad .$$

Now is it necessary to implement the primary constraints in the theory. In order to do this, we introduce a set of Lagrange multipliers $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and λ_5 in the canonical Hamiltonian. So we arrive in the so called Dirac Hamiltonian

$$H = H_c + (\lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3 + \lambda_4 V_4 + \lambda_5 V_5). \quad (16)$$

The fundamental Poisson brackets are

$$[A_0, \pi_0] = \delta(x - y), \quad [A_i, \pi_j] = \delta_{ij} \delta(x - y), \quad (17)$$

and

$$[\phi_i, \Pi_j] = \delta_{ij} \delta(x - y), \quad [b, \pi_b] = \delta(x - y), \quad (18)$$

the others being zero. The time evolution of the primary constraints with the Dirac Hamiltonian (16) determines all the multiplier fields. Namely,

$$\dot{V}_1 = [V_1, \int d^2x H] = [\pi_0, \int d^2x H_c] + \lambda_5 \approx 0, \quad (19)$$

and

$$\dot{V}_2 = [V_2, \int d^2x H] = [\phi \cdot \phi - 1, \int d^2x H_c] = \mathbf{\Pi} \cdot \phi \approx 0. \quad (20)$$

There is no more secondary constraints engendered by the consistency condition for V_3 e V_4 , since $[V_3, V_4] \neq 0$. The Lagrange multiplier λ_5 is obtained from (19). On the other hand, the constraint $V_6 = \mathbf{\Pi} \cdot \phi \approx 0$ will give us more one secondary constraint, namely

$$\dot{V}_6 = [V_6, \int d^2x H] = [\mathbf{\Pi} \cdot \phi, \int d^2x H_c] + \lambda_2 \approx 0. \quad (21)$$

However the consistency condition will determine the Lagrange multiplier λ_2 , and we will have not any further secondary constraints in the theory. Therefore our set of fully second-class constraints are

$$V_1 = \pi_0 \approx 0, \quad (22)$$

$$V_2 = \phi \cdot \phi - 1 \approx 0, \quad (23)$$

$$V_3 = \pi_1 + \frac{\kappa}{2} A_2 \approx 0, \quad (24)$$

$$V_4 = \pi_2 - \frac{\kappa}{2} A_1 \approx 0, \quad (25)$$

$$V_5 = \pi_b + A_0 \approx 0, \quad (26)$$

$$V_6 = \phi \cdot \mathbf{\Pi} \approx 0. \quad (27)$$

Note that, there is not first-class constraints in the theory. This fact is due we choose to work with a theory in a covariant gauge-fixed form. As is it known the second-class constraints must be eliminated since they do not generate physical transformations. In order to do this we introduce the Dirac brackets defined as

$$[\Theta(x), \theta(y)]_D = [\Theta(x), \theta(y)] - [\Theta(x), V_i(\zeta)](C^{-1})_{ij} [V_j(\xi), \theta(y)] \quad (28)$$

where $C_{ij}(x, y)$ is an invertible matrix defined by

$$C_{ij} = [V_i(x), V_j(y)]. \quad (29)$$

The C_{ij} in this case has the form

$$C(x, y) = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & \kappa & 0 & 0 \\ 0 & 0 & -\kappa & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \end{bmatrix} \delta(x - y). \quad (30)$$

and its inverse is written as

$$(C)^{-1}(x, y) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{\kappa} & 0 & 0 \\ 0 & 0 & -\frac{1}{\kappa} & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix} \delta^{-1}(x - y). \quad (31)$$

Therefore the set of nonvanishing Dirac brackets is given below

$$[A_0, b]_D = \delta(x - y) \quad (32)$$

$$[A_i, \pi_j]_D = \frac{1}{2} \delta_{ij} \delta(x - y) \quad (33)$$

$$[A_i, A_j]_D = \frac{\varepsilon_{ji}}{\kappa} \delta(x - y) \quad (34)$$

$$[\pi_i, \pi_j]_D = \frac{\kappa}{4} \varepsilon_{ji} \delta(x - y) \quad (35)$$

$$[\phi_i, \Pi_j]_D = (\delta_{ij} - \phi_i(x) \phi_j(y)) \delta(x - y) \quad (36)$$

$$[\Pi_i, \Pi_j]_D = (\phi_j(y) \Pi_i(x) - \phi_i(x) \Pi_j(y)) \delta(x - y) \quad (37)$$

Quantization follows in the usual way by replacing $i[F, G]_D \longrightarrow [\hat{F}, \hat{G}]$, where \hat{F} and \hat{G} denote operators. Then we obtain

$$[\hat{A}_0, \hat{b}] = i \delta(x - y) \quad (38)$$

$$[\hat{A}_i, \hat{\pi}_j] = \frac{i}{2} \delta_{ij} \delta(x - y) \quad (39)$$

$$[\hat{A}_i, \hat{A}_j] = \frac{i}{\kappa} \varepsilon_{ji} \delta(x - y) \quad (40)$$

$$[\hat{\pi}_i, \hat{\pi}_j] = \frac{i\kappa}{4} \varepsilon_{ji} \delta(x - y) \quad (41)$$

$$[\hat{\phi}_i, \hat{\Pi}_j] = i \left(\delta_{ij} - \hat{\phi}_i(x) \hat{\phi}_j(y) \right) \delta(x - y) \quad (42)$$

$$[\hat{\Pi}_i, \hat{\Pi}_j] = i \left(\hat{\phi}_j(y) \hat{\Pi}_i(x) - \hat{\phi}_i(x) \hat{\Pi}_j(y) \right) \delta(x - y). \quad (43)$$

III. THE NON-MINIMAL GAUGED O(3) NONLINEAR SIGMA MODEL

Now we construct a nonminimal version of the gauged O(3) sigma model described by the Lagrangian

$$L = \frac{1}{2} \nabla_\mu \phi \cdot \nabla^\mu \phi + \frac{\kappa}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - A_\mu \partial^\mu b + \frac{\alpha}{2} b^2, \quad (44)$$

where the minimal gauge covariant derivative was changed by

$$\nabla_\mu \phi = \partial_\mu \phi + \left[eA_\mu + \frac{g}{2} \epsilon_{\mu\nu\sigma} F^{\nu\sigma} \right] \mathbf{n} \times \phi. \quad (45)$$

Here $F^{\mu\nu}$ stands for the field-strength of the gauge field, the Levi-Civita symbol $\epsilon_{\mu\nu\lambda}$ is fixed by $\epsilon_{012} = 1$ and g is the nonminimal coupling constant.

Proceeding closely to the formulation for the minimal case we obtain the canonical momenta conjugate to the fields $A_0, A_k, b, \phi_1, \phi_2, \phi_3$ respectively, namely

$$\pi_0 = 0 \quad (46)$$

$$\pi^k = \frac{\kappa}{2} \epsilon^{ki} A_i + g \epsilon^{kj} \mathbf{n} \times \phi \cdot D_j \phi + g^2 F^{0k} |\mathbf{n} \times \phi|^2 \quad (47)$$

$$\pi_b = -A_0 \quad (48)$$

$$\Pi_1 = \dot{\phi}_1 - eA_0 \phi_2 - \frac{g}{2} \epsilon_{ij} F^{ij} \phi_2 \quad (49)$$

$$\Pi_2 = \dot{\phi}_2 + eA_0 \phi_1 + \frac{g}{2} \epsilon_{ij} F^{ij} \phi_1 \quad (50)$$

$$\Pi_3 = \partial_0 \phi_3. \quad (51)$$

Note that $D_j \phi = \partial_j \phi + eA_j \mathbf{n} \times \phi$. The primary constraints for the Lagrangian (44) are

$$V_1 = \pi_0 \approx 0 \quad (52)$$

$$V_2 = \phi \cdot \phi - 1 \approx 0 \quad (53)$$

$$V_3 = \pi_b + A_0 \approx 0 \quad (54)$$

Therefore the canonical Hamiltonian can be written as

$$\begin{aligned} H_c = & \frac{1}{2} \frac{1}{g^2 |\mathbf{n} \times \phi|^2} [\pi^k \pi_k + \kappa \epsilon_{jk} \pi^k A_j + \frac{\kappa^2}{4} A_j A^j + \kappa g A_j \mathbf{n} \times \phi \cdot D^j \phi \\ & + g^2 (\mathbf{n} \times \phi \cdot D^j) (\phi \times \phi \cdot D_j \phi) + 2g \pi^k \epsilon_{jk} \mathbf{n} \times \phi \cdot D^j \phi] \\ & + g \epsilon_{kj} \partial^k A_0 \mathbf{n} \times \phi \cdot D^j \phi + A_j \partial^j b - \frac{\alpha}{2} b^2 - \frac{\kappa}{2} \epsilon^{ij} A_0 \partial_i A_j + \frac{\mathbf{\Pi} \cdot \mathbf{\Pi}}{2} \\ & + \left(eA_0 + \frac{g}{2} \epsilon_{ij} F^{ij} \right) \cdot (\phi_2 \Pi_1 - \phi_1 \Pi_2) + \pi^k \partial_k A_0 - \frac{1}{2} D^j \phi \cdot D_j \phi \\ & + \frac{1}{4} g^2 \left[\frac{1}{2} [\epsilon_{ij} F^{ij}]^2 - F_{ij} F^{ij} \right] (\phi_1^2 + \phi_2^2). \end{aligned} \quad (55)$$

Now we address the issue of the existence of secondary and tertiary constraints in the model. We note that the consistency conditions $\dot{V}_1 \approx 0$ e $\dot{V}_3 \approx 0$ leads us to the Lagrange multipliers λ_3 e λ_1 respectively. On the other hand, the consistency condition $\dot{V}_2 \approx 0$ give us a secondary constraint (V_4) and $\dot{V}_4 \approx 0$ provide the Lagrange multiplier λ_1 . There is no more secondary (or tertiary) constraints, therefore our final set of constraints is

$$V_1 = \pi_0 \approx 0 \quad (56)$$

$$V_2 = \phi \cdot \phi - 1 \approx 0 \quad (57)$$

$$V_3 = \pi_b + A_0 \approx 0 \quad (58)$$

$$V_4 = \mathbf{\Pi} \cdot \boldsymbol{\phi} \approx 0 \quad (59)$$

These constraints are clearly of second-class since

$$[V_1, V_3] = -\delta(x - y) \quad (60)$$

$$[V_2, V_4] = 2\boldsymbol{\phi}(\mathbf{x}) \cdot \boldsymbol{\phi}(\mathbf{y})\delta(x - y). \quad (61)$$

This is not unexpected since the Lagrangian (44) is gauge fixed as in the previous case. It is worth mentioning that the simpler structure of constraints (56-59) is due to the presence of derivative terms in the canonical momenta conjugate to the fields A_k , ϕ_1 , ϕ_2 , represented by the Eqs. (47), (49) and (50), respectively.

The matrix $C(x, y)$ in this case has the form

$$C(x, y) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} \delta(x - y). \quad (62)$$

Using the inverse of the matrix (62), we determine the nonvanishing Dirac brackets

$$[A_0, b]_D = \delta(x - y) \quad (63)$$

$$[A_i, \pi_j]_D = \delta_{ij}\delta(x - y) \quad (64)$$

$$[\phi_i, \Pi_j]_D = (\delta_{ij} - \phi_i(x)\phi_j(y))\delta(x - y) \quad (65)$$

$$[\Pi_i, \Pi_j]_D = (\phi_j(y)\Pi_i(x) - \phi_i(x)\Pi_j(y))\delta(x - y) \quad (66)$$

Consequently, the quantized theory is obtained through the commutators

$$[\hat{A}_0, \hat{b}] = i\delta(x - y) \quad (67)$$

$$[\hat{A}_i, \hat{\pi}_j] = i\delta_{ij}\delta(x - y) \quad (68)$$

$$[\hat{\phi}_i, \hat{\Pi}_j] = i\left(\delta_{ij} - \hat{\phi}_i(x)\hat{\phi}_j(y)\right)\delta(x - y) \quad (69)$$

$$[\hat{\Pi}_i, \hat{\Pi}_j] = i\left(\hat{\phi}_j(y)\hat{\Pi}_i(x) - \hat{\phi}_i(x)\hat{\Pi}_j(y)\right)\delta(x - y) \quad (70)$$

Here we would like to call attention to a comparison between the commutator relations of two cases. Note that in the nonminimal case the dependence of the Chern-Simons coefficient k disappears.

IV. CONCLUSIONS

In summary, we have first investigated some effects on the symplectic structure of the gauged $O(3)$ nonlinear sigma model with Chern-Simons term due to introducing of a nonminimal Pauli coupling. To begin with, we consider the model with a minimal coupling and a covariant gauge fixing and after we treat the nonminimal version. We have used the Dirac quantization formalism in order to accomplish the quantization of the models. We found that the symplectic structure was changed by the introduction of the Pauli coupling. Indeed, the reduction of the number of constraints made the quantization procedure simpler. Furthermore the commutator relations of the nonminimal case seems to indicate that the Chern-Simons term plays no role in the quantized theory and consequently leads to a no existence of fractional spin in the theory. Nevertheless, this issue requires more studies. It is worthwhile to mentioning that results in an Abelian Chern-Simons-Higgs model coupled non-minimally to matter fields seems to show the presence of fractional spin¹⁵, even in the absence of the Chern-Simons term in the theory. Therefore, it would be interesting to investigating the possibility of fractional spin and statistics in this model. As a matter of fact, this has been accomplished by Lee *et al.*¹⁶ in a slightly different context, but in the minimal case. So far, to the best of our knowledge, no study was accomplished for the nonminimal case.

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